The Bose-Nelson Sorting Problem for 11 and 12 Inputs

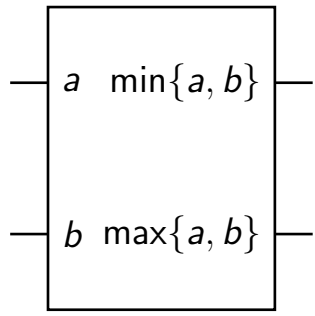
Jannis Harder

July 7, 2020

Overview

Comparator Networks and Sorting Networks

- Circuit of *comparator* gates



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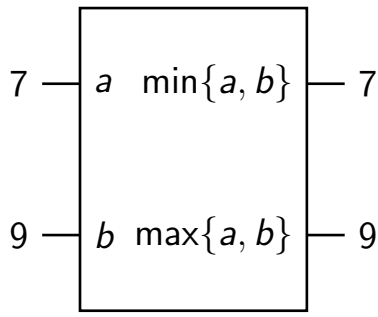

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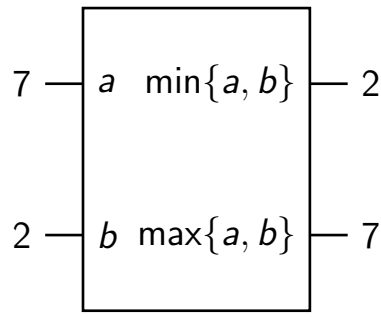

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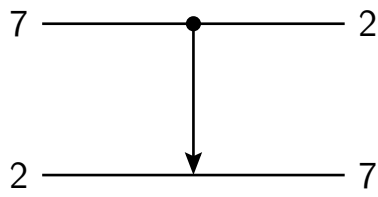

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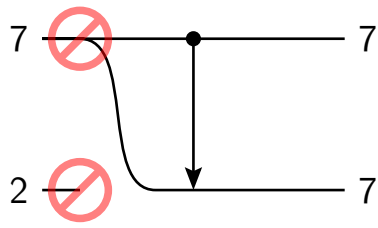

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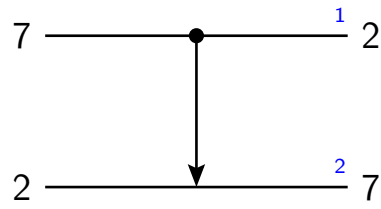

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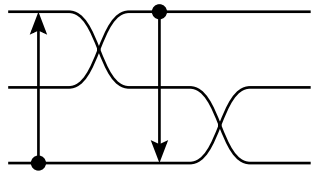

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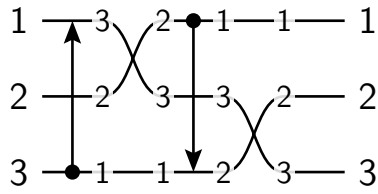

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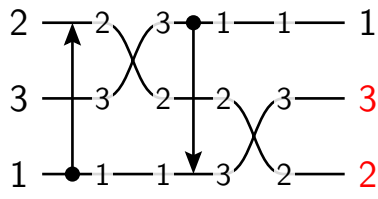

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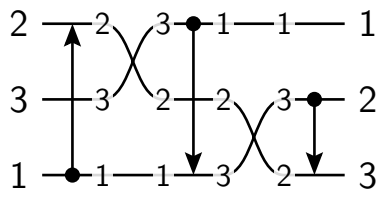

Comparator Networks and Sorting Networks

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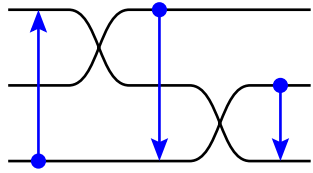
- The *size* is the number of comparators



The Bose-Nelson Sorting Problem

What is the minimum size s_n of a sorting network with n inputs?

Timeline of Selected Results

- Information theory bound: $s_n \geq \lceil \log_2 n! \rceil$, $\Omega(n \log n)$.

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- 2014 Codish, Cruz-Filipe, Frank & Schneider-Kamp: Best known networks with 9 and 10 inputs are optimal.

Known bounds for small n (OEIS A003075)

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
s_n	0	1	3	5	9	12	16	19	25	29	33	37	41	45
											⋮	⋮	⋮	⋮
											35	39	45	51

Upper bounds from TAOCPv3

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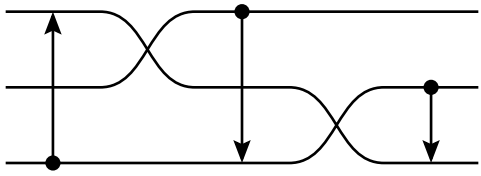
Sorting Network Preliminaries

Algebraic Representation

- Sequence of operations on length-n sequences



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- Sequence of operations on length-n sequences
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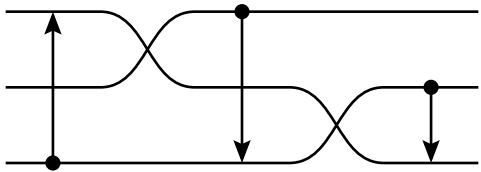
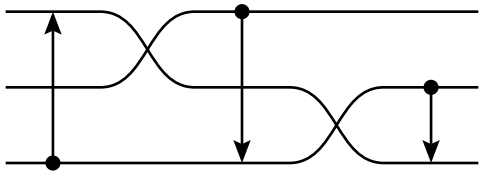


$[3,1]\ (1,2)\ [1,3]\ (2,3)\ [2,3]$

Algebraic Representation

- Sequence of operations on length-n sequences
- Comparator $[i, j]$ and exchange (i, j) operations
- Written as postfix from left to right



$[3,1]\ (1,2)\ [1,3]\ (2,3)\ [2,3]$

$$(3,2,1)\ [3,1](1,2)[1,3](2,3)[2,3] = (1,2,3)$$

Standard Forms

- *Permuted standard form*: $i < j$ for all comparators $[i, j]$ and all comparators precede all exchanges

$$[1,2][2,3](1,2) \checkmark \quad [1,2][3,2](1,2) \times \quad [1,2](1,2)[2,3] \times$$

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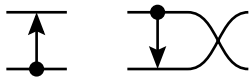
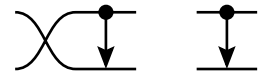
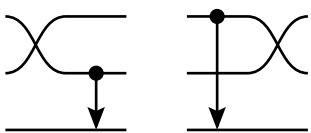


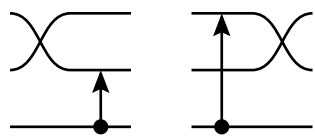
$[j,i] = (i,j)[i,j]$



$[i,j](i,j) = [i,j]$



$(i,j)[j,k] = [i,k](i,j)$



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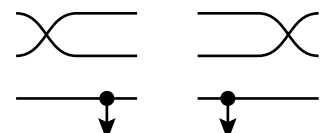

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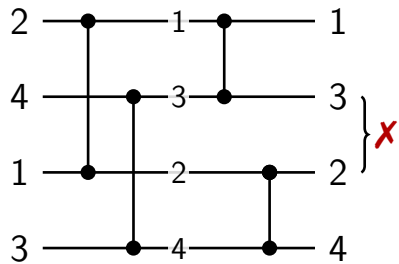
Zero-One Principle

- A comparator network is a sorting network if and only if it sorts any Boolean input sequence (Floyd & Knuth 1973)



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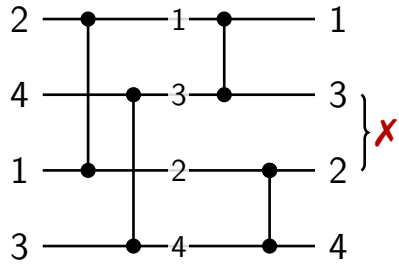

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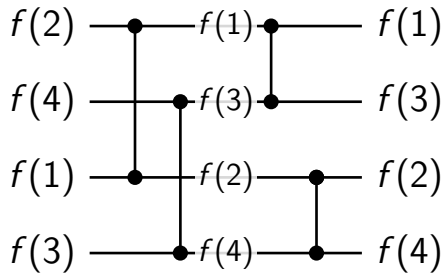

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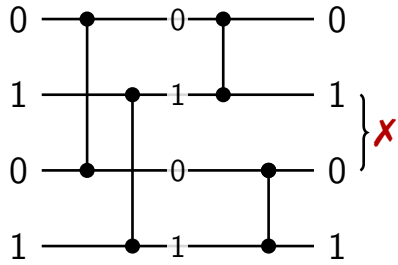

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- $s(X)$ is the minimum size over all partial sorting networks on X
- Let B_n be the set of all length-n Boolean sequences, then $s(B_n) = s_n$

Monotonicity, Symmetries & Successors

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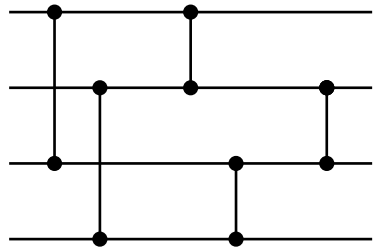
Van Voorhis' Bound 1/2

- Sorting network C of size s_n



C

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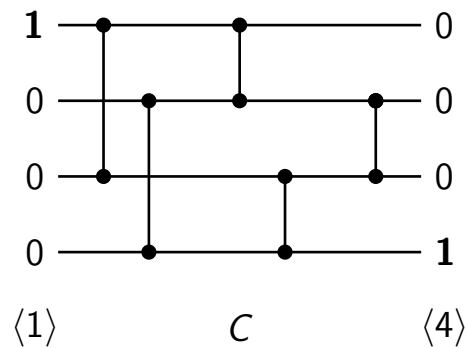

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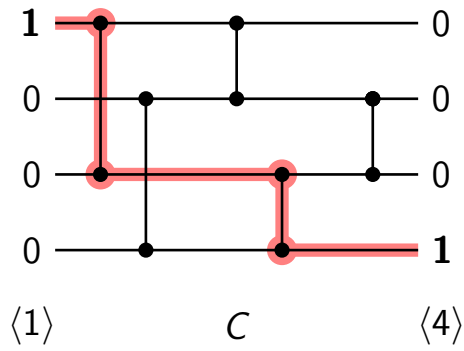

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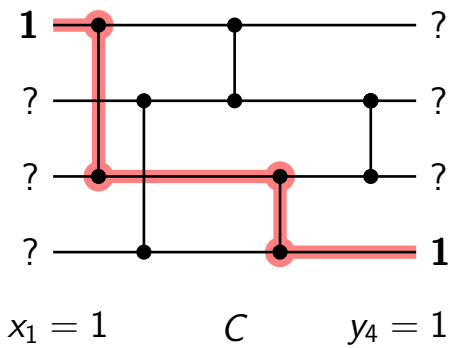

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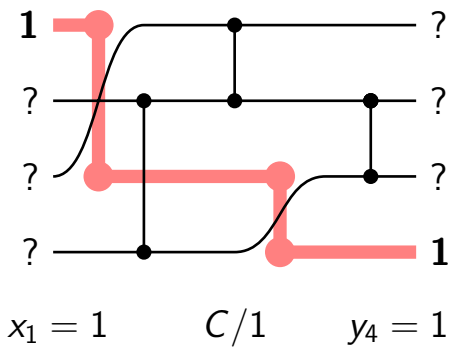

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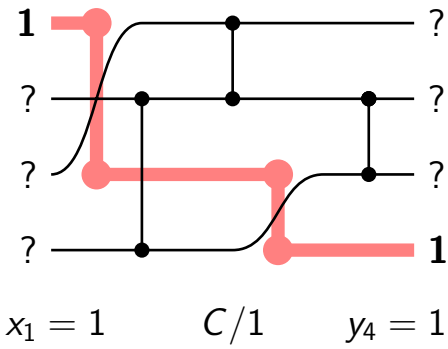
- Resulting network has $d_i(C)$ fewer comparators and sorts $n - 1$ inputs



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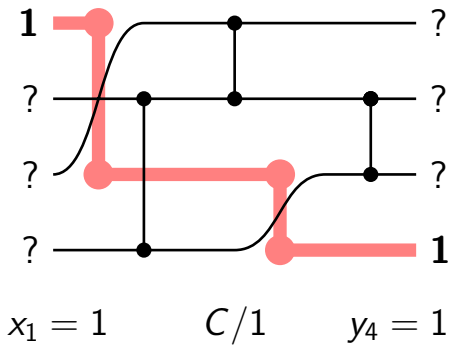
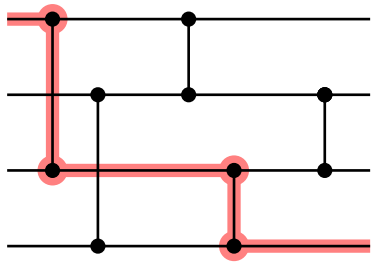
- $s_n \geq d_i(C) + s_{n-1}$



Van Voorhis' Bound 2/2

- Combining bounds for each channel: $s_n \geq \max_i d_i(C) + s_{n-1}$



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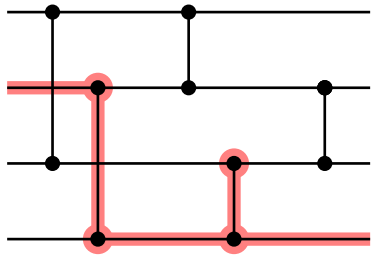

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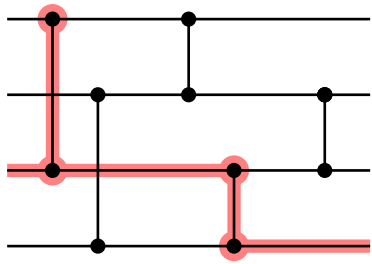

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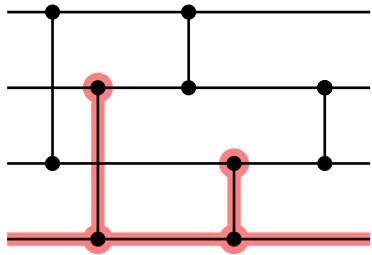

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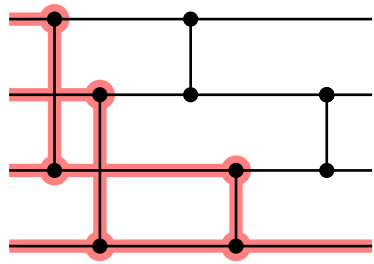

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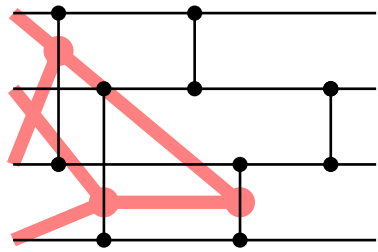

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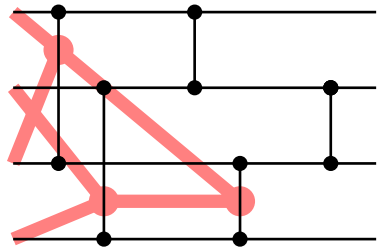

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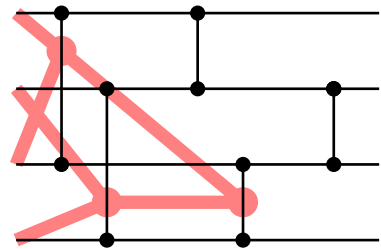

C

New Bound on Partial Sorting Networks

Generalizing Van Voorhis' Bound?

n	1	2	3	4	5	6	7	8	9	10	11	12
s_n	0	**1**	**3**	**5**	9	**12**	16	**19**	25	**29**	35	**39**
		↑	↑	↑		↑		↑		↑		↑

- Van Voorhis' bound is strict for more than half of the known values!
- Can we generalize it to partial sorting networks?

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  E.g. $B_n/i = B_{n-1}$, $\{000, 100, 010, 011, 111\}/2 = \{00, 01, 11\}$

  ✗ ✗ ✓ ✓ ✓

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- If C sorts X, C/i sorts X/i for prunable i of X

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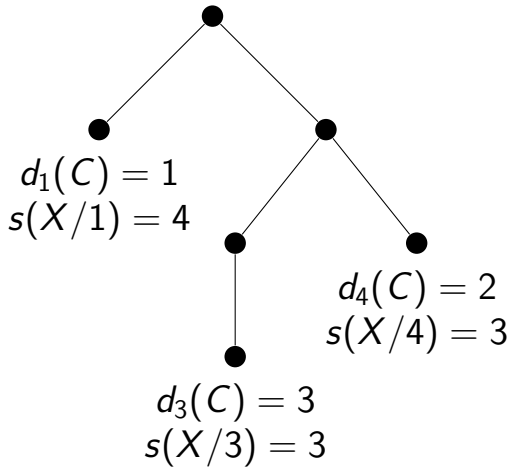

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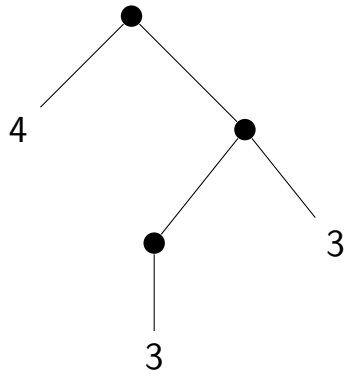

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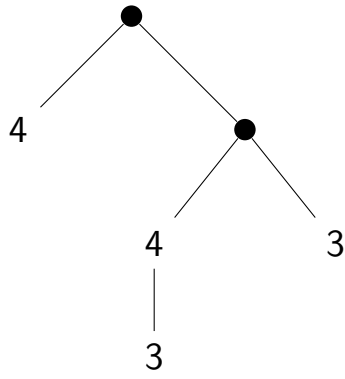

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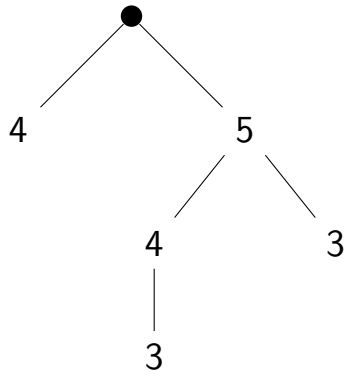

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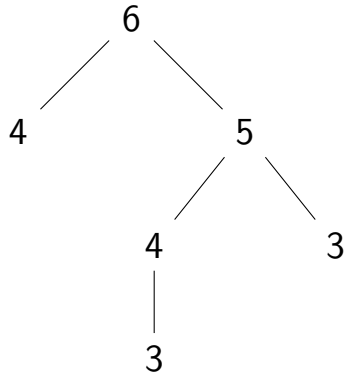

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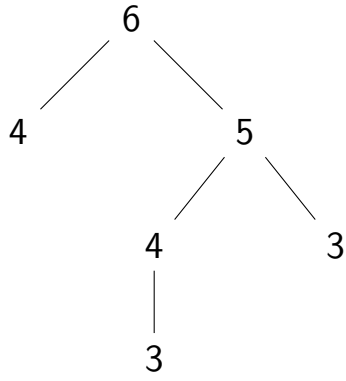
- Minimize over *all* trees with leaves $s(X/i)$



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- Finds minimum root value $H_\star(v_1, \ldots, v_k)$ by repeatedly combining the two smallest values

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- $s(X) \geq H_{\diamond}(s(X/i_1), \ldots, s(X/i_k))$ for prunable $i_1, \ldots, i_k$ of X

Computer Search

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- Improve bounds until $b(X) = \{s(X)\}$

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- Less memory, more overlapping subproblems
- Represent X as labeled graph and use canonicalization
- Very efficient in practice (nauty & Traces by McKay & Piperno)

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- Computing $s_9 = 25$, $s_{10} = 29$ takes less than 5 s and 40 MB of RAM on my laptop

Formal Verification

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- Resulting certificate is below 3 GB and contains 12 659 079 steps

Formalization of the Problem

Formalization of s_n using Isabelle/HOL:

type-synonym *vect* = ‹*nat* ⇒ *bool*›

definition *fixed-width-vect* :: ‹*nat* ⇒ *vect* ⇒ *bool*› **where**
‹*fixed-width-vect n v* = (∀ *i* ≥ *n*. *v i* = *True*)›

type-synonym *cmp* = ‹*nat* × *nat*›

definition *apply-cmp* :: ‹*cmp* ⇒ *vect* ⇒ *vect*› **where**
‹*apply-cmp c v* = (
let (*a*, *b*) = *c*
in v(
a := *min* (*v a*) (*v b*),
b := *max* (*v a*) (*v b*)))›

definition *lower-size-bound* :: ‹*vect set* ⇒ *nat* ⇒ *bool*› **where**
‹*lower-size-bound V b* = (∀ *cn*. (∀ *v* ∈ *V*. *mono* (*fold apply-cmp cn v*)) ⟶ *length cn* ≥ *b*)›

definition *lower-size-bound-for-width* :: ‹*nat* ⇒ *nat* ⇒ *bool*› **where**
‹*lower-size-bound-for-width w b* = *lower-size-bound* {*v*. *fixed-width-vect w v*} *b*›

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- Running the checker on the 11-input certificate prints `Just (11,35)` and took 3 h and below 8 GB on my laptop

- Sourcecode & formal proof: `github.com/jix/sortnetopt`
- Twitter: `@jix_`
- Email: `me@jix.one`
- Slides: `files.jix.one/s/20200707.pdf`

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